

ON THE STABILITY LIMIT CHANGE DUE TO IMPERFECTIONS

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ABSTRACT

The minimum volume stability limit of axisymmetric liquid bridges has been obtained in the past for zero Bond number, no solid body rotation and equal disks. When gravity, rotation or different disk diameters are considered, only a few numerical results are available and the singular case of small values of these parameters have been only partly considered.

An analytical study using the Lyapunov-Schmidt method considering small values of gravity, disk diameter ratio and rotation rate is presented and the variation of the stability limit determined.

INTRODUCTION

The fluid configuration considered here consists of an isothermal mass of liquid of volume V held by surface tension forces between two parallel coaxial solid disks (of radii R_1 and R_2 , respectively) placed a distance L apart (Figure 1). Such fluid configuration can be uniquely defined by the following dimensionless parameters: the dimensionless volume $V = V/(\pi R_0^2 L)$, where $R_0 = (R_1 + R_2)/2$, the slenderness $\Lambda = L/(2R_0)$, the dimensionless disk radii difference, $H = (R_2 - R_1)/(R_2 + R_1)$, the Bond number $B = \Delta\rho g R_0^2/\sigma$ (where $\Delta\rho$ is the difference in densities between the liquid bridge and the surrounding medium, g the acceleration acting on the liquid bridge and σ stands for the surface tension) and the Weber number $W = \Delta\rho \Omega^2 R_0^3/\sigma$.

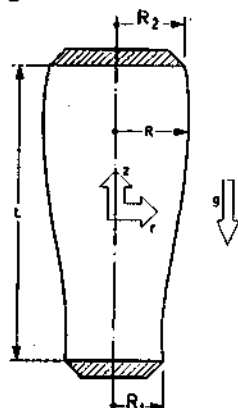


Fig. 1. Geometry and coordinate system for the liquid bridge problem.

The stable axisymmetric equilibrium shapes can lose their stability with respect to either axisymmetric or non-axisymmetric perturbations (a method to determine this limit can be found in /1/ and more recent results have been reported on the influence of Bond number /2/, the Weber number /3/ and disk diameter inequality /4/). All these studies obtain a formulation which is finally solved numerically in order to compute the stability limit. Other studies /5/ use a formulation in terms of elliptic functions to obtain the minimum volume stability limit when neither gravity nor rotation is considered (although arbitrary disk diameter difference could be considered) and, in other cases numerical methods have been used to compute the stability limits /6,7/. In most of the papers dealing with liquid bridges only axisymmetric configurations have been considered. It is difficult to determine from the above mentioned studies the behaviour for small values of B ,

W and H . Analytical studies have been done in the neighbourhood of the cylinder /8,9/ but there is little knowledge of this behaviour but for the particular case of the cylinder.

This paper deals with the analysis of the stability limits of liquid bridges when small values of Bond number, Weber number and disk inequality are considered. The method of Lyapunov-Schmidt is used to determine the behaviour close to the well-known (see f.i. /5/) minimum volume stability limit for $B = W = H = 0$.

PROBLEM FORMULATION

Equilibrium shapes of liquid bridges are described by the Young-Laplace equation, which in dimensionless variables reads

$$M(F) - Bz + \frac{1}{2}WF^2 + P = \frac{F_{zz}}{(1+F_z^2)^{3/2}} - \frac{1}{F(1+F_z^2)^{1/2}} - Bz + \frac{1}{2}WF^2 + P = 0 \quad (1)$$

This equation is to be integrated with the anchoring conditions and that of volume conservation

$$F(\pm\Lambda) = 1 \pm H \quad , \quad V = \frac{1}{2\Lambda} \int_{-\Lambda}^{\Lambda} F^2 dz \quad (2)$$

To write down the above expressions, all lengths have been made dimensionless with R_0 . P is a constant related with the pressure level which has been made dimensionless with σ / R_0 .

LINEAR STABILITY

Critical points result after linearization of the above formulation [8,9]. It is well-known that the solution of the system (1)-(2) in the case $B=W=0$, in parametric form using the variable $u = F(a, \varphi)$ is given by

$$\left. \begin{aligned} z(a, \varphi) &= \sqrt{\alpha} [F(a, \varphi) \cos a + E(a, \varphi) + b] = \sqrt{\alpha} [u \cos a + E(a, \operatorname{am} u) + b] \\ F_e(a, \varphi) &= \sqrt{\alpha} \sqrt{1 - \sin^2 a \sin^2 \varphi} = \sqrt{\alpha} \operatorname{dn} u \quad , \quad P_e = 2\alpha^{-1/2} (1 + \cos a)^{-1} \end{aligned} \right\} \quad \begin{aligned} \varphi_1 \leq \varphi \leq \varphi_2 \\ u_1 \leq u \leq u_2 \end{aligned} \quad (3)$$

where $-\pi/2 \leq a \leq \pi/2$ and b are constants and $F(a, \varphi)$ and $E(a, \varphi)$ are the elliptic integrals of first and second kind. If $H=0$ boundary conditions yield $\sin \varphi_1 = \sin \varphi_2$ ($\operatorname{dn} u_1 = \operatorname{dn} u_2$). The introduction of the expansions $F(z) = F_e(z(\varphi)) + \varepsilon f(z) + O(\varepsilon^2)$ and $P = P_e + \varepsilon p + O(\varepsilon^2)$ where ε stands for the magnitude of the deformation of the interface, will allow us to calculate $f(z)$ after neglecting $O(\varepsilon^2)$ terms in the problem formulation. The resulting problem is then

$$M'[F_e](f) + p = \frac{1}{F_e} \frac{d}{dz} \left[F_e (1 + F_e^2)^{-3/2} \frac{df}{dz} \right] + F_e^{-2} (1 + F_e^2)^{-1/2} f + p = 0 \quad , \quad (4)$$

$$f(z = \Lambda) = f(z = -\Lambda) = 0 \quad , \quad \int_{-\Lambda}^{\Lambda} F_e f dz = 0 \quad (5)$$

Non trivial solutions of the problem appear only for a certain combination of the parameters, where the transition from stable to unstable equilibrium shapes occurs. The general solution of the differential equation (4) is

$$\begin{aligned} f(z(u)) &= A \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn}^2 u + \cos a} + B \frac{\operatorname{dn} u (2 \operatorname{sn}^2 u - 1) + \operatorname{sn} u \operatorname{cn} u [(1 + \cos^2 a) u - 2 E(a, \operatorname{am} u)]}{\operatorname{dn}^2 u + \cos a} \\ &+ Q \frac{\operatorname{dn} u \operatorname{sn}^2 u + \operatorname{sn} u \operatorname{cn} u [u - E(a, \operatorname{am} u)]}{\operatorname{dn}^2 u + \cos a} \quad , \end{aligned} \quad (6)$$

where $z(u)$ is given by equation (3) and the values of A , B and Q are to be found by imposing the boundary conditions (5). Non trivial solutions for A , B and Q appear if $F_e(\pm\Lambda) = 0$. This is known to be the relevant instability if $\Lambda > 2.128$ [1,5]. For having $F_e(\pm\Lambda) = 0$ an integer number of periods of the Plateau curve should be taken ($u_1 + u_2 = 2nK(a)$ or $u_1 - u_2 = 2nK(a)$, n being an integer). Nevertheless, only the case $n=1$ must be considered as it is the first instability that appears. In this case, if $V < 1$, $0 < a < a_1 < \frac{\pi}{2}$, $\alpha = 1$ and if $V > 1$, $0 < a < \frac{\pi}{2}$, $\alpha = 1 / \cos^2 a$ and Λ and V_e are given as functions of a

$$\Lambda = \alpha^{1/2} [\cos a K(a) + E(a)] \quad , \quad V_e = \alpha \left[\frac{2}{3} \frac{(1 + \cos a)^2 E(a)}{K(a) \cos a + E(a)} - \frac{\cos a}{3} \right] \quad (7)$$

A plot of this stability limit can be found in [5]. Note that for $\Lambda < 2.128$ the instability considered here is no longer the relevant one.

BIFURCATION TO EQUILIBRIUM STATES

Let $g(z)$ and q the expressions representing higher order terms. The new expansions for F and P are then $F(z) = F_e(z) + \varepsilon f(z) + g(z)$ and $P = P_e + \varepsilon p + q$ which, after substitution in equations (1)-(2) gives the new formulation

$$M(F_e(z) + \varepsilon f(z) + g(z)) - Bz + \frac{1}{2} W(F_e(z) + \varepsilon f(z) + g(z))^2 + P_e + \varepsilon p + q = 0 \quad , \quad (8)$$

$$g(\pm\Lambda) = \pm H, \quad \frac{1}{2\Lambda} \int_{-\Lambda}^{\Lambda} \left[2g(z)(F_e(z) + \mathcal{E}f(z)) + g(z)^2 \right] dz = v \equiv V - V_e, \quad \int_{-\Lambda}^{\Lambda} g(z)f(z)dz = 0. \quad (9)$$

The last (additional) condition is required in order to uniquely define the parameter ε . The problem (8)–(9) allows us to calculate q and $g(z)$ in terms of Λ , v , H , B and W . As these parameters are assumed to be small enough, calculations can be performed by using standard perturbation techniques. It is known that this procedure requires the anticipation of certain properties of the solution, situation which can be avoided by using the idea of the bifurcation equation. In this case, instead of (8) the equation to be solved is:

$$M(F_e(z) + \mathcal{E}f(z) + g(z)) - Bz + \frac{1}{2}W(F_e(z) + \mathcal{E}f(z) + g(z))^2 + P_e + \varepsilon p + q + \Psi f(z) = 0 \quad (10)$$

and expressions (9)–(10) uniquely define $g(z; \varepsilon, v, H, B, W)$, $q(z; \varepsilon, v, H, B, W)$, $\Psi(\varepsilon, v, H, B, W)$, which can be represented with expansions in the form $r = \delta_1 r_1 + \delta_2 r_2 + \delta_3 r_3 + \dots$, where r stands for either g , q or Ψ and equal indexes implies summation, at least in a neighbourhood of $\varepsilon = v = H = B = W = 0$. In these expressions $\delta_1 = \varepsilon$, $\delta_2 = v$, $\delta_3 = H$, $\delta_4 = B$, $\delta_5 = W$. Such solutions will correspond to the solution of original set of equations (8)–(9) if and only if the parameters involved satisfy $\Psi(\varepsilon, v, H, B, W) = 0$ which is called the bifurcation equation.

Before solving the problem it is convenient to analyze the symmetries involved in the problem which will allow us to anticipate some characteristics of the solution and to ease the algebra involved [9]. As it can be seen, the problem is invariant under the following set of symmetries: $z \rightarrow -z$; $H \rightarrow -H$, $\varepsilon \rightarrow -\varepsilon$, $B \rightarrow -B$, $\Psi \rightarrow -\Psi$. From this symmetry it is deduced that $\Psi(\varepsilon, v, H, B, W) = -\Psi(-\varepsilon, v, -H, -B, W)$. It can be deduced without any further calculation that the only first-order coefficients which are non-zero are those corresponding to the terms in H and in B , and the second-order coefficients to be taken into account are those in εv and εW . Once the above mentioned non-zero terms are taken into account the only third-order term that can be relevant is that in ε^3 . Thus the expansion of Ψ can be simplified to yield $\Psi = \Psi_3 H + \Psi_4 B + 2\Psi_{12}\varepsilon v + 2\Psi_{15}\varepsilon W + \Psi_{111}\varepsilon^3 + \dots$ where it has been taken into account that $\Psi_{ij} = \Psi_{ji}$, and $\Psi_{ijk} = \Psi_{jik} = \Psi_{kji}$. Setting $\Psi = 0$ the original problem is recovered and this equation can define the value of ε as a function of the remaining parameters v , H , B and W . Note that the term $\Psi_3 H + \Psi_4 B$ can be zero. In that case, higher order terms, in particular those in vH , vB , $\varepsilon^2 H$, $\varepsilon^2 B$, εv^2 , $\varepsilon^3 v$ and ε^5 should be considered.

The relevant coefficients can be deduced after a set of linear problems following a procedure similar to that described by Meseguer et al. [9]. Thus, equation $\Psi = 0$ can be written as

$$0 = (1 - \alpha \cos a)(1 + \alpha \cos a)^{-1} (1 - \sqrt{\alpha \cos a})^{-2} \left[\tilde{\varepsilon}^3 \phi_{111} + \tilde{\varepsilon} v \phi_{12} + \tilde{\varepsilon} W \phi_{15} + H \phi_3 + B \phi_4 \right] \quad (11)$$

where $\tilde{\varepsilon} = \varepsilon(1 - \alpha \cos a)/(1 + \alpha \cos a)$ and

$$\phi_{111} = \alpha^{-1/2} \frac{2}{\pi(1 - \alpha \cos a)^2} \left[\frac{(K(a) - \alpha E(a))^2}{K(a) - 2\alpha E(a)} - (1 + \alpha^2 \cos^2 a)K(a) + 2\alpha E(a) \right] \quad (12)$$

$$\phi_{12} = -\frac{\alpha^{1/2}}{K(a) - 2\alpha E(a)} \frac{4(K(a)\cos a + E(a))(K(a) - \alpha E(a))}{\pi(1 - \alpha \cos a)(1 + \alpha \cos a)^2} \quad (13)$$

$$\phi_{15} = \alpha^{-1/2} \frac{1 + \alpha \cos a}{6\pi(1 - \alpha \cos a)} \left\{ (K(a) - \alpha E(a)) \left[3 + 4\alpha^2 \cos^2 a + \frac{\alpha E(a)}{K(a) - 2\alpha E(a)} \right] - 3 \left[(1 + \alpha^2 \cos^2 a)K(a) - 2\alpha^3 E(a) \cos^2 a \right] \right\} \quad (14)$$

$$\phi_3 = -\frac{4}{\pi} \frac{1}{1 + \alpha \cos a}, \quad \phi_4 = \frac{2}{\pi} \frac{K(a)\cos a + E(a)}{(1 - \cos a)} |1 - v| \quad (15)$$

The coefficients ϕ_{111} , ϕ_{12} , ϕ_{15} , ϕ_3 , ϕ_4 have been plotted in Figure 2. As can be seen from

equation (11) the effect of having small values of B and H shifts the stability limit in an amount proportional to the two-thirds power of the quantity $\varphi_3 H + \varphi_4 B$. This shift can be of course zero for special combinations of H and B . In this case the symmetry of the problem broken by the imperfections is recovered and the shifting in the stability limit will be given by the not considered higher order terms. The values of B and H ($H/B = -\varphi_4/\varphi_3$) for this compensation to appear are plotted in Figure 3.

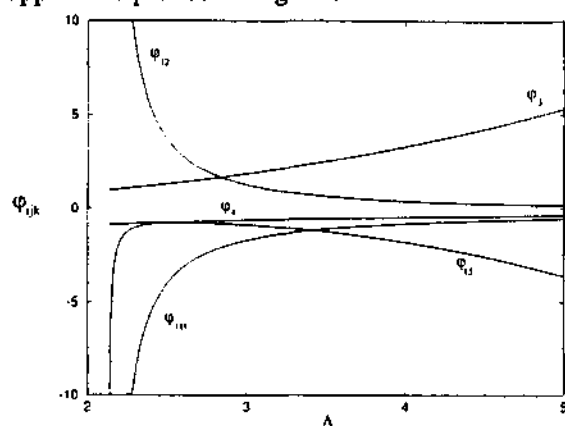


Fig. 2. Bifurcation equation coefficients

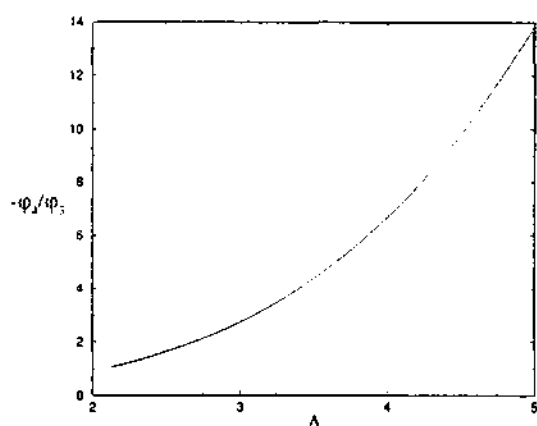


Fig. 3. Value of the ratio $H/B = -\varphi_4/\varphi_3$ for not having imperfections.

CONCLUSIONS

A theoretical expression for the stability limit of long liquid bridges close to the limit for $H = B = W = 0$ has been obtained and the effect of having small values of disk diameter difference, Bond number and rotation rate considered. It has been shown that the influence of Bond number and disk inequality is as the two-thirds power of its value and that both effects can cancel each other (for particular values). On the other hand, the effect of a small rotation rate consists of linearly shifting the stability limit.

ACKNOWLEDGMENTS

This work has been supported by the Spanish Comisión Interministerial de Ciencia y Tecnología (CICYT), Project No. ESP92-0001-CP.

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